Particular Solutions to the Time-Fractional Heat Equation

Simon Kelow
Northern Arizona University

Mentor: Kevin Hayden

What is Fractional Calculus?

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Thus fractional calculus extends the derivative operator into a continuous operator.

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Inital Conditon: $u(0,x) = \frac{-4a}{L^2}x^2 + \frac{4a}{L}x$

Solutions: PDE vs FDE

PDE:

$$u(t,x) = \sum_{n=0}^{\infty} \frac{32a}{\pi^3(2n+1)^3} \sin\left(\frac{(2n+1)\pi}{L}x\right) e^{\frac{-(2n+1)^2\pi^2}{L^2}t}$$

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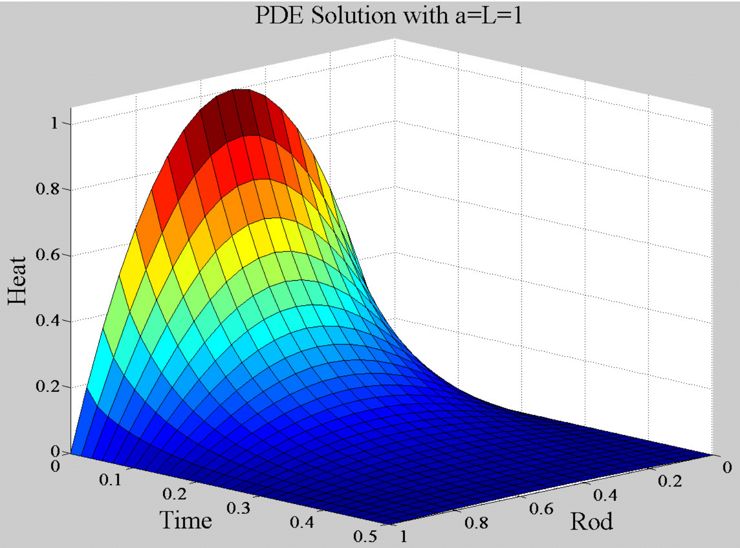
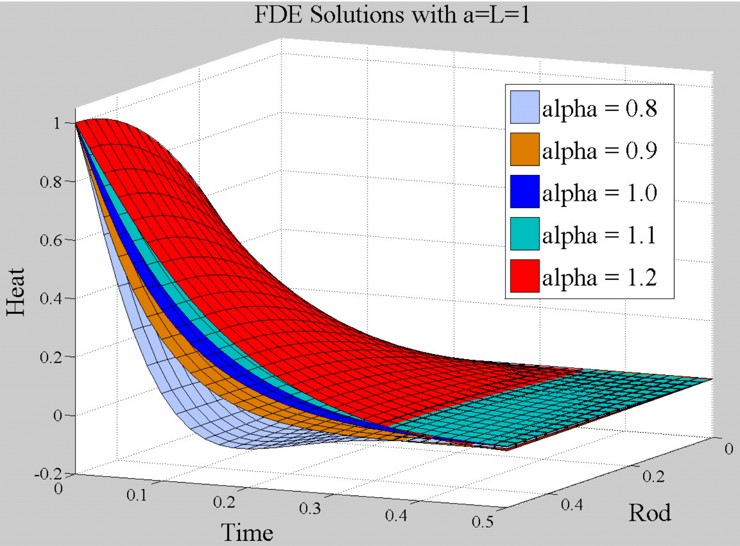
FDE:

$$u(t,x) = \sum_{n=0}^{\infty} \frac{32a}{\pi^3(2n+1)^3} \sin\left(\frac{(2n+1)\pi}{L}x\right) e^{\sqrt[\alpha]{\frac{-(2n+1)^2\pi^2}{L^2}}t}$$

Heat Equation: PDE vs FDE



Heat Equation: PDE vs FDE



Thank You. Any Questions?

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